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Variations on a strange semi-continuity result

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Abstract

This paper is devoted to the study of some lower semi-continuity results whose model example is

$$\liminf_{n\to+\infty}\int_\Omega \nabla u_n\cdot\nabla v_n \geqslant \int_\Omega \nabla u\cdot\nabla v,$$

for any sequence (u_n, v_n) which weakly converges to (u, v) in $H_0^1(\Omega)^2$ and satisfies the positivity assumptions $-\Delta u_n \geqslant 0$ and $-\Delta v_n \geqslant 0$. Extensions of this semi-continuity result are obtained in the following cases: two fixed linear operators, two monotone operators, two varying linear operators, varying domains and non-uniformly bounded operators. In these various cases, the operators under consideration are second-order elliptic operators in divergence form and truncation arguments play an essential role. Finally, the case of higher-order operators is treated by a quite different approach.

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1. Introduction

The model of the results that we prove in the present paper is the following:

Theorem 1.1. *Let Ω be a bounded open set of $\mathbb{R}^d$, $d \geqslant 1$. Consider two sequences u_n and v_n* in *$H_0^1(\Omega)$ such that*

$$u_n \rightharpoonup u \quad \text{and} \quad v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \tag{1.1}$$

$$-\Delta u_n \geqslant 0 \quad \text{and} \quad -\Delta v_n \geqslant 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.2}$$

Then for any function $\psi \in C^\infty(\overline{\Omega})$, $\psi \geqslant 0$, we have

$$\liminf_{n \to +\infty} \int_\Omega \psi\, \nabla u_n \cdot \nabla v_n \geqslant \int_\Omega \psi\, \nabla u \cdot \nabla v. \tag{1.3}$$

At the first glance, this result seems a little bit strange. And the proof that we present below is even more strange. On the one hand, the proof of the result is not "symmetric'' in u_n and v_n, even if the assumptions of Theorem 1.1 are symmetric. Indeed, an important step of the proof will be to prove that

$$\left.\begin{array}{l} v_n \rightharpoonup v \quad \text{weakly in} \quad H_0^1(\Omega) \\ -\Delta\, v_n \geqslant 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \end{array}\right\} \quad \Longrightarrow \quad (v_n - v)^- \to 0 \quad \text{strongly in } H_0^1(\Omega). \tag{1.4}$$

Moreover in spite of appearances, this result has nothing to do with Fatou's lemma. Of course, when $\psi = 1$, one has

$$\int_\Omega \nabla u_n \cdot \nabla v_n = \langle -\Delta u_n, v_n \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{1.5}$$

One can prove that the sequence $\nabla u_n \cdot \nabla v_n$ converges almost everywhere to $\nabla u \cdot \nabla v$, but in general the function $\nabla u_n \cdot \nabla v_n$ is not nonnegative (see a counterexample in Example 2.8 in Section 2.1). On the other hand, since the assumptions imply that $v_n \geqslant 0$, one has $-\Delta u_n\, v_n \geqslant 0$ in $\mathcal{D}'(\Omega)$, but the sequence $-\Delta u_n\, v_n$ does not converge almost everywhere. So the two terms of (1.5) enjoy one but not both assumptions of Fatou's lemma.

The origin of the present paper has to be found in the paper [20] of the second author who proved that, in a Dirichlet space $\mathcal{H}$, the assertion

$$(1.1) + (1.2) \quad \Longrightarrow \quad (1.3) \quad \text{with} \quad \psi = 1$$

is equivalent to the fact that the truncation is weakly continuous in $\mathcal{H}$, i.e.

$$u_n \rightharpoonup u \quad \text{weakly in } \mathcal{H} \quad \Longrightarrow \quad u_n^+ \rightharpoonup u^+ \quad \text{weakly in } \mathcal{H},$$

a fact which itself is strongly related to a compactness property of Rellich's type.

In the present paper, we do not treat the question of this equivalence, but we present a series of variations of the above Theorem 1.1. In these variations, we consider problems of increasing complexity.

We begin with the case of "fixed'' operators:

- the case where in (1.2), $-\Delta$ is replaced by two operators:

$$-\operatorname{div}(A(x)\nabla u_n) \geqslant 0 \quad \text{and} \quad -\operatorname{div}(B(x)\nabla v_n) \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{1.6}$$

  where the two matrices A and B have bounded coefficients and where B is coercive (Section 2.1);
- or even by two monotone operators (Section 2.2):

$$-\operatorname{div}(a(x, \nabla u_n)) \geqslant 0 \quad \text{and} \quad -\operatorname{div}(b(x, \nabla v_n)) \quad \text{in } \mathcal{D}'(\Omega);$$

- finally, we remark that case (1.6) can be generalized to an "abstract'' setting in a Dirichlet space (Remark 2.14).

Then we pass to the case of "varying'' operators or domains:

- the case where (1.6) is replaced by

$$-\operatorname{div}(A_n(x)\nabla u_n) \geqslant 0 \quad \text{and} \quad -\operatorname{div}(B_n(x)\nabla v_n) \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{1.7}$$

  where A_n and B_n are two equi-coercive and equi-bounded sequences of matrices which H-converge (Section 3);
- the case where the operators are fixed, but where the open sets vary:

$$\begin{cases} u_n \in H^1_0(\Omega_n) \quad \text{and} \quad v_n \in H^1_0(\Omega_n), \\ -\Delta u_n \geqslant 0 \quad \text{and} \quad -\Delta v_n \geqslant 0 \quad \text{in } \mathcal{D}'(\Omega_n), \end{cases}$$

  where Ω_n is obtained by removing from Ω many small holes, which leads to the appearance of a "strange term" in the limit operator (Section 4.2);
- the case where the domain is fixed, but where the operators, given by (1.7), are varying with $B_n = A_n$ a sequence of symmetric, equi-coercive, but no more equi-bounded matrices, which leads to nonlocal effects (Section 4.3);
- actually, the two latest cases are applications of an abstract "degenerated'' general framework (Section 4.1).

Finally, we consider the case of higher-order operators, where the operator $-\Delta$ is replaced by a fourth-order operator (for instance Δ^2) in the whole space $\mathbb{R}^d$ (Section 5.1), i.e.

$$\begin{cases} u_n \rightharpoonup u \quad \text{and} \quad v_n \rightharpoonup v \quad \text{weakly in} \quad H^2(\mathbb{R}^d), \\ \Delta^2 u_n \geqslant 0 \quad \text{and} \quad \Delta^2 v_n \geqslant 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d), \end{cases}$$

or even by a m-order operator $(-\Delta)^m$, for any integer $m \geqslant 2$ (Section 5.2).

In these two latest cases, the proofs are of course quite different from the proofs used in Sections 2–4, where truncations of functions in $H_0^1(\Omega)$ (or in $W_0^{1,p}(\Omega)$) play an essential role.

2. The case of two fixed operators

2.1. The linear case

Let Ω be a bounded open set of $\mathbb{R}^d$. For $0 < \alpha \leqslant \beta < +\infty$, we denote by $\mathcal{M}(\alpha, \beta; \Omega)$ the set of the measurable matrix-valued functions $A : \Omega \longrightarrow \mathbb{R}^{d\times d}$ such that

$$\text{a.e.} \quad x \in \Omega, \quad \forall \xi \in \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geqslant \alpha\,|\xi|^2 \text{ and } A(x)^{-1}\xi \cdot \xi \geqslant \beta^{-1}\,|\xi|^2. \tag{2.1}$$

We have the following result:

Theorem 2.1. *Let $A \in L^\infty(\Omega)^{d\times d}$ and $B \in \mathcal{M}(\alpha, \beta; \Omega)$. Let u_n and v_n be two sequences in $H_0^1(\Omega)$ such that*

$$u_n \rightharpoonup u \quad \textit{and} \quad v_n \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega), \tag{2.2}$$

$$-\operatorname{div}(A\nabla u_n) \geqslant 0 \quad \textit{and} \quad -\operatorname{div}(B\nabla v_n) \geqslant 0 \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.3}$$

Then we have the strong convergence

$$(v_n - v)^- \longrightarrow 0 \quad \textit{strongly in } H_0^1(\Omega), \tag{2.4}$$

and the following inequality holds true:

$$\forall \psi \in C^\infty(\overline{\Omega}), \ \psi \geqslant 0, \quad \liminf_{n\to+\infty} \int_\Omega \psi A\nabla u_n \cdot \nabla v_n \geqslant \int_\Omega \psi\, A\nabla u \cdot \nabla v. \tag{2.5}$$

Remark 2.2. Note that the assumption $A \in \mathcal{M}(\alpha, \beta; \Omega)$ and even the fact that $A \geqslant 0$ is not necessary for Theorem 2.1 to hold. It is sufficient to assume that $B \in \mathcal{M}(\alpha, \beta; \Omega)$ and that $A \in L^\infty(\Omega)^{d\times d}$.

Remark 2.3. Inequality (2.5) of Theorem 2.1 still holds if the positivity assumptions (2.3) are replaced by

$$\begin{aligned} &-\operatorname{div}(A\nabla u_n) \geqslant F_n \quad \text{and} \quad -\operatorname{div}(B\nabla v_n) \geqslant G_n \quad \text{in } \mathcal{D}'(\Omega),\\ &\text{where} \quad F_n \longrightarrow F \quad \text{and} \quad G_n \longrightarrow G \quad \text{strongly in } H^{-1}(\Omega). \end{aligned}$$

This result is obtained by the same proof as that of Theorem 2.1.

Remark 2.4. In general, the sequence u_n in Theorem 2.1 does not strongly converge to u in $H_0^1(\Omega)$ as shown by the following example:

Example 2.5. Let T_n be the truncature at height n, i.e. $T_n(t) := \max(-n, \min(t, n))$ for $t \in \mathbb{R}$. Let Ω be the unit ball of $\mathbb{R}^d$, $d \geqslant 2$, $A = B := I_d$ (the identity matrix) and let u_n be defined by

$$u_n(x) := \begin{cases} \frac{1}{\sqrt{n}} T_n\left(\frac{1}{|x|^{d-2}} - 1\right) & \text{if} \quad d \geqslant 3, \\ \frac{1}{\sqrt{n}} T_n(-\ln|x|) & \text{if} \quad d = 2. \end{cases} \tag{2.6}$$

It is easy to check that u_n weakly converges to 0 in $H_0^1(\Omega)$ but not strongly. Moreover, since T_n is a Lipschitz continuous increasing and concave function on $[0, +\infty[$, and since the function $\frac{1}{|x|^{d-2}}$ (respectively $-\ln|x|$) is harmonic in $\mathbb{R}^d \setminus \{0\}$, one has $-\Delta u_n \geqslant 0$ in $\mathcal{D}'(\Omega)$.

Remark 2.6. In general, inequality (2.5) is strict. For example, take $\psi = 1$, $u_n = v_n$ and u_n as in Example 2.5.

Remark 2.7. Inequality (2.5) is reminiscent to Fatou's lemma applied to the sequence $A\nabla u_n \cdot \nabla v_n$: indeed, by the compactness result of [24], the weak convergence of u_n to u in $H_0^1(\Omega)$ and the nonnegativity of $-\operatorname{div}(A\nabla u_n)$ imply that u_n actually strongly converges to u in $W_0^{1,q}(\Omega)$ for any $q < 2$, whence (up to a subsequence) ∇u_n converges to ∇u a.e. in Ω and $A\nabla u_n \cdot \nabla u_n$ converges to $A\nabla u \cdot \nabla v$ a.e. in Ω. However, one cannot apply Fatou's lemma since the function $A\nabla u_n \cdot \nabla v_n$ is not nonnegative in general, as shown by the following example:

Example 2.8. Let Ω be the unit ball of $\mathbb{R}^2$ and $A = B := I_d$. Let u_n defined by (2.6) and let w be defined by

$$w(x_1, x_2) := \left(1 - x_1^2 - x_2^2\right) e^{x_2}.$$

Then $w \in C^\infty(\overline{\Omega}) \cap H_0^1(\Omega)$ and one easily checks that $-\Delta w = \left(x_1^2 + x_2^2 + 4x_2 + 3\right) e^{x_2}$ is nonnegative on Ω, while for every $z = (0, z_2) \in \Omega$ with $0 < z_2 < \sqrt{2} - 1$,

$$\nabla w(z) \cdot z = \left(1 - 2z_2 - z_2^2\right) z_2\, e^{z_2} > 0.$$

Then the sequences u_n and $v_n := u_n + w$ satisfy assumptions (2.2) and (2.3) with $u = 0$ and $v = w$, and one has for every $x \in \Omega$ such that $|x| > e^{-n}$,

$$\nabla u_n(x) \cdot \nabla v_n(x) = \frac{1}{n} \frac{1}{|x|^2} - \frac{1}{\sqrt{n}} \frac{\nabla w(x) \cdot x}{|x|^2},$$

which is negative in the neighbourhood of any $z = (0, z_2)$ with $0 < z_2 < \sqrt{2} - 1$, when n is large enough. So, one does not have $\nabla u_n \cdot \nabla v_n \geqslant 0$ a.e. in Ω.

Remark 2.9. For any $\psi \in C^\infty(\overline{\Omega})$, we have

$$\int_\Omega \psi\, A\nabla u_n \cdot \nabla v_n = \int_\Omega A\nabla u_n \cdot \nabla(\psi v_n) - \int_\Omega v_n\, A\nabla u_n \cdot \nabla\psi$$

and by Rellich's compactness theorem,

$$\int_\Omega v_n\, A\nabla u_n \cdot \nabla\psi \underset{n\to+\infty}{\longrightarrow} \int_\Omega v\, A\nabla u \cdot \nabla\psi.$$

Therefore, inequality (2.5) is equivalent to

$$\liminf_{n\to+\infty} \langle -\operatorname{div}(A\nabla u_n), \psi v_n\rangle \geqslant \langle -\operatorname{div}(A\nabla u), \psi v\rangle, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Observe that the equivalence between (2.5) and (2.7) holds true whenever $A \in L^\infty(\Omega)^{d\times d}$ without assuming that $A \in \mathcal{M}(\alpha, \beta; \Omega)$.

Inequality (2.7) is reminiscent to some Fatou's lemma type for distributions since $-\operatorname{div}(A\nabla u_n) \geqslant 0$ and $\psi v_n \geqslant 0$ in $\mathcal{D}'(\Omega)$ (note that $v_n \geqslant 0$ is an immediate consequence of the maximum principle). However, the sole assumptions

$$u_n \rightharpoonup u \quad\text{and}\quad v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \tag{2.8}$$

$$-\operatorname{div}(A\nabla u_n) \geqslant 0 \quad\text{and}\quad v_n \geqslant 0 \quad\text{in}\quad \mathcal{D}'(\Omega), \tag{2.9}$$

are not sufficient to obtain (2.7), as shown by the following example:

Example 2.10. The present example is derived from the model example of homogenization in perforated domains studied by Cioranescu and Murat [11] (Model Example 2.1).

Let $d \geqslant 3$ and let Ω be an open cube of $\mathbb{R}^d$. For any $n \geqslant 1$, let w_n be the function of $H^1(\Omega)$ defined by $w_n := 0$ in any ball B_{k,r_n} centered on $\frac{1}{n}k$, $k \in \mathbb{Z}^d$, and of radius $r_n := n^{-\frac{d}{d-2}}$, $w_n := 1$ outside the union of the balls $B_{k,\frac{1}{n}}$ with the same centers and radius $\frac{1}{n}$, and w_n is harmonic in any annulus $B_{k,\frac{1}{n}} \setminus \overline{B_{k,r_n}}$. It is proved in [11] that

$$w_n \rightharpoonup 1 \ \text{ weakly in } \ H^1(\Omega) \quad\text{and}\quad -\Delta w_n = \mu_n - \gamma_n \ \text{ in } \ \mathcal{D}'(\Omega),$$

where μ_n is a positive Radon measure carried by the spheres $\partial B_{k,\frac{1}{n}}$ and where γ_n is a positive Radon measure carried by the spheres $\partial B_{k,r_n}$; it is also proved that γ_n weakly converges in $H^{-1}(\Omega)$ to a positive constant γ.

Let u_n be the solution of $u_n \in H_0^1(\Omega), -\Delta u_n = \gamma_n$ in $\mathcal{D}'(\Omega)$, and for a fixed $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$, let $v_n := \varphi w_n$. It is easy to check that u_n weakly converges to u in $H_0^1(\Omega)$, where $u \in H_0^1(\Omega), -\Delta u = \gamma$, that v_n weakly converges to $v = \varphi$ in $H_0^1(\Omega)$, that $-\Delta u_n \geqslant 0$ and that $v_n \geqslant 0$ in Ω.

Since $w_n = 0$ in B_{k,r_n} for any $k \in \mathbb{Z}^d$, we have, for any $\psi \in C_c^\infty(\Omega)$, $\psi \geqslant 0$,

$$\langle -\Delta u_n, \psi v_n \rangle = \langle \gamma_n, \psi \varphi w_n \rangle = 0.$$

However,

$$\langle -\Delta u, \psi v \rangle = \langle \gamma, \psi \varphi \rangle = \gamma \int_\Omega \psi \varphi > 0 \quad \text{if} \quad \int_\Omega \psi \varphi \neq 0.$$

This proves that (2.7) does not follow from (2.8) and (2.9); in other terms, in Theorem 2.1, assertion $-\operatorname{div}(B\nabla v_n) \geqslant 0$ in $\mathcal{D}'(\Omega)$ cannot be replaced by the (weaker) assertion $v_n \geqslant 0$ in Ω.

Remark 2.11. The fact that (2.7) does not follow from (2.8) and (2.9) is due to a lack of compactness. Actually, by adding some extra compactness we obtain the following lower semi-continuity result, whose proof is based on the same ingredients as the proof of Theorem 2.1:

Proposition 2.12. *Let σ_n be a sequence in $L^2(\Omega)^d$ such that*

$$\sigma_n \rightharpoonup \sigma \ \text{weakly in } L^2(\Omega)^d \quad \text{and} \quad -\operatorname{div}(\sigma_n) \geqslant 0 \ \text{in } \mathcal{D}'(\Omega) \tag{2.10}$$

and let v_n be a sequence in $H_0^1(\Omega)$ such that, for some $v \in H_0^1(\Omega)$,

$$v_n \longrightarrow v \ \text{strongly in} \ L^2(\Omega) \quad \text{and} \quad (v_n - v)^- \longrightarrow 0 \ \text{strongly in} \ H_0^1(\Omega). \tag{2.11}$$

Then we have

$$\forall \psi \in C^\infty(\overline{\Omega}), \ \psi \geqslant 0, \quad \liminf_{n \to +\infty} \int_\Omega \psi\, \sigma_n \cdot \nabla v_n \geqslant \int_\Omega \psi\, \sigma \cdot \nabla v. \tag{2.12}$$

Example 2.13. Consider u_n and w_n defined in Example 2.10, the sequences $\sigma_n := \nabla u_n$ and $v_n := \varphi w_n$, and $\sigma := \nabla u$, $v = \varphi$, with $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$. Then σ_n and v_n satisfy the assumptions of Proposition 2.12, except the second assumption of (2.11) since $(v_n - v)^- = \varphi(1 - w_n)$ does not strongly converge to 0 in $H_0^1(\Omega)$. Therefore, the strong convergence of $(v_n - v)^-$ in $H_0^1(\Omega)$ is necessary in order to obtain inequality (2.12).

Proof of Theorem 2.1.

Step 1: Proof of the strong convergence (2.4).

As v_n and v belong to $H_0^1(\Omega)$, so is the case for $(v_n - v)^+$ and $(v_n - v)^-$. On the one hand, using the equality $\nabla(v_n - v)^+ \cdot \nabla(v_n - v)^- = 0$ a.e. in Ω and the coerciveness of B, we have

$$\begin{aligned} -\int_\Omega B\nabla(v_n - v)\cdot\nabla(v_n - v)^- &= \int_\Omega B\nabla(v_n - v)^-\cdot\nabla(v_n - v)^- \\ &\geqslant \alpha\int_\Omega |\nabla(v_n - v)^-|^2. \end{aligned}$$

On the other hand, $-\operatorname{div}(B\nabla v_n) \geqslant 0$ implies that

$$\begin{aligned} -\int_\Omega B\nabla(v_n - v)\cdot\nabla(v_n - v)^- &= -\langle -\operatorname{div}(B\nabla v_n), (v_n - v)^-\rangle \\ &\quad + \int_\Omega B\nabla v\cdot\nabla(v_n - v)^- \\ &\leqslant \int_\Omega B\nabla v\cdot\nabla(v_n - v)^-. \end{aligned}$$

Therefore, we obtain

$$\alpha\int_\Omega |\nabla(v_n - v)^-|^2 \leqslant \int_\Omega B\nabla v\cdot\nabla(v_n - v)^-.$$

Since $(v_n - v)^-$ weakly tends to 0 in $H_0^1(\Omega)$, the right hand side tends to 0. Thus $\nabla(v_n - v)^-$ strongly converges to 0 in $L^2(\Omega)^d$, which implies the strong convergence (2.4).

Step 2: Proof of (2.5).

As said before in Remark 2.9, inequality (2.5) is equivalent to (2.7) that we will prove now. Let $\psi \in C_c^\infty(\Omega)$, $\psi \geqslant 0$. Since $-\operatorname{div}(A\nabla u_n) \geqslant 0$, we have

$$\begin{aligned} \langle -\operatorname{div}(A\nabla u_n), \psi v_n\rangle &= \langle -\operatorname{div}(A\nabla u_n), \psi(v_n - v)^+ - \psi(v_n - v)^-\rangle \\ &\quad + \int_\Omega A\nabla u_n\cdot\nabla(\psi v) \\ &\geqslant \langle -\operatorname{div}(A\nabla u_n), -\psi(v_n - v)^-\rangle + \int_\Omega A\nabla u_n\cdot\nabla(\psi v). \end{aligned}$$

By the first step, the sequence $(v_n - v)^-$ strongly converges to 0 in $H_0^1(\Omega)$ whence

$$\langle -\operatorname{div}(A\nabla u_n), -\psi(v_n - v)^-\rangle \underset{n\to+\infty}{\longrightarrow} 0.$$

Moreover, the weak convergence of ∇u_n to ∇u in $L^2(\Omega)^d$ implies that

$$\int_\Omega A\nabla u_n \cdot \nabla(\psi v) \underset{n\to+\infty}{\longrightarrow} \int_\Omega A\nabla u \cdot \nabla(\psi v) = \langle -\operatorname{div}(A\nabla u)\,,\, \psi v\rangle.$$

This establishes (2.7) and thus (2.5). $\square$

Observe that in the above proof we only used the assumption $A \in L^\infty(\Omega)^{d\times d}$ and not $A \in \mathcal{M}(\alpha, \beta; \Omega)$ (see Remark 2.2).

Remark 2.14. Theorem 2.1 can be easily extended with the same proof to the following general framework of Dirichlet spaces (see [20] for another proof):

Let $\mathcal{H}$ be an Hilbert space and $\|\cdot\|$ be its norm. Let a and b be two bilinear forms from $\mathcal{H}\times\mathcal{H}$ into $\mathbb{R}$ which satisfy the following properties:
(i) a and b are continuous, i.e. there exists $c > 0$ such that

$$\forall\, u, v \in \mathcal{H}, \quad |a(u, v)| \leqslant c\,\|u\|\,\|v\| \quad \text{and} \quad |b(u, v)| \leqslant c\,\|u\|\,\|v\|,$$

(ii) b is coercive, i.e. there exists $\alpha > 0$ such that

$$\forall\, u \in \mathcal{H}, \quad b(u, u) \geqslant \alpha\,\|u\|^2.$$

We also assume that there exists a (nonlinear) operator T from $\mathcal{H}$ into $\mathcal{H}$ which satisfies the following properties:
(j) T is even, i.e.

$$\forall u \in \mathcal{H}, \quad T(-u) = T(u),$$

(jj) T is bounded, i.e. there exists $c > 0$ such that

$$\forall\, u \in \mathcal{H}, \quad \|T(u)\| \leqslant c\,\|u\|,$$

(jjj) T is sequentially weakly continuous on $\mathcal{H}$, i.e.

$$w_n \rightharpoonup w \text{ weakly in } \mathcal{H} \implies T(w_n) \rightharpoonup T(w) \text{ weakly in } \mathcal{H},$$

$(jjjj)$ the operators defined by $T^\pm(u) := \frac{1}{2}\,(T(u) \pm u)$ satisfy

$$T \circ T^\pm = T^\pm \quad \text{and} \quad \forall\, u \in \mathcal{H}, \;\; b(T^+(u), T^-(u)) \leqslant 0.$$

Let $\mathcal{H}^+$ be the positive cone of $\mathcal{H}$ defined by $\mathcal{H}^+ := \{u \in \mathcal{H} \,|\, T^-(u) = 0\}$. Then we have the following result:

Theorem 2.15. *Let u_n and v_n be two sequences in $\mathcal{H}$ such that*

$$u_n \rightharpoonup u \quad \textit{and} \quad v_n \rightharpoonup v \quad \textit{weakly in } \mathcal{H}, \tag{2.13}$$

$$\forall\, w \in \mathcal{H}^+, \quad a(u_n, w) \geqslant 0 \quad \textit{and} \quad b(v_n, w) \geqslant 0. \tag{2.14}$$

Then we have the strong convergence

$$T^-(v_n - v) \;\longrightarrow\; 0 \quad \textit{strongly in } \mathcal{H}, \tag{2.15}$$

and the following inequality holds true:

$$\liminf_{n \to +\infty} \; a(u_n, v_n) \geqslant a(u, v). \tag{2.16}$$

The following result allows us to introduce test functions in the lower semi-continuity inequality (2.16):

Theorem 2.16. *Under the assumptions of Theorem* 2.15, *assume moreover that there exists a set $\mathcal{D}$ of "multipliers" with $\mathcal{D} \subset \mathcal{H}$, which satisfies*

$$\forall\, \varphi \in \mathcal{D}, \quad u \longmapsto \varphi u \textit{ is a linear bounded operator on } \mathcal{H}.$$

Then the following extension of inequality (2.16) *holds true*:

$$\forall\, \psi \in \mathcal{D} \cap \mathcal{H}^+, \quad \liminf_{n \to +\infty} \; a(u_n, \psi v_n) \geqslant a(u, \psi v).$$

Remark 2.17. The result of Theorem 2.16 can be extended in the following way. Let $M : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear bounded operator which is nonnegative, i.e. such that $Mw \in \mathcal{H}^+$ for any $w \in \mathcal{H}^+$. Then, under the assumptions of Theorem 2.15 we have

$$\liminf_{n \to +\infty} \; a(u_n, Mv_n) \geqslant a(u, Mv). \tag{2.17}$$

Indeed, since $w_n := v_n - v + T^-(v_n - v) = T^+(v_n - v)$ belongs to the cone $\mathcal{H}^+$ (because $T^- \circ T^+ = 0$) so does Mw_n, whence by positivity (2.14) of a and by its bilinearity we obtain

$$a(u_n, Mv_n) \geqslant a(u_n, Mv) - a\left(u_n, M(T^-(v_n - v))\right).$$

Hence, the weak convergence (2.13) of u_n and the strong convergence (2.15) of $T^-(v_n - v)$ imply inequality (2.17).

Example 2.18. Set $\mathcal{H} := H_0^1(\Omega)$, $T(u) := |u|$, $\mathcal{D} := C_c^\infty(\Omega)$ and for any $u, v \in H_0^1(\Omega)$,

$$a(u, v) := \int_\Omega A\nabla u \cdot \nabla v \quad \text{and} \quad b(u, v) := \int_\Omega B\nabla u \cdot \nabla v,$$

where A belongs to $L^\infty(\Omega)^{d\times d}$ and B to $\mathcal{M}(\alpha, \beta; \Omega)$. Then Theorem 2.1 (completed by Remark 2.2) reads as a particular case of Theorems 2.15 and 2.16.

2.2. The monotone case

Let Ω be a bounded open set of $\mathbb{R}^d$, let $p \in]1, +\infty[$ and $p' := \frac{p}{p-1}$. We consider two functions a, b from $\Omega \times \mathbb{R}^d$ into $\mathbb{R}^d$ which satisfy the following properties:

(i) a and b are Carathéodory functions, i.e.

$$\begin{cases} \text{a.e. } \ x \in \Omega, \ a(x, \cdot) \ \text{and} \ \ b(x, \cdot) \ \ \text{are continuous on} \ \ \mathbb{R}^d, \\ \forall\, \xi \in \mathbb{R}^d, \quad a(\cdot, \xi) \ \ \text{and} \ \ b(\cdot, \xi) \ \ \text{are measurable on} \ \ \Omega, \end{cases}$$

(ii) a and b have $(p-1)$-growth in the following sense: there exists a positive constant c and a function γ in $L^p(\Omega)$ such that

$$\text{a.e. } \ x \in \Omega, \ \forall\, \xi \in \mathbb{R}^d, \quad |a(x, \xi)| + |b(x, \xi)| \leqslant c\, (|\gamma(x)| + |\xi|)^{p-1},$$

(iii) a and b are coercive, i.e. there exists a positive constant α and a function δ in $L^1(\Omega)$ such that

$$\text{a.e. } \ x \in \Omega, \ \text{for any } \ \xi \in \mathbb{R}^d, \quad \begin{cases} a(x, \xi) \cdot \xi \geqslant \alpha|\xi|^p - |\delta(x)|, \\ b(x, \xi) \cdot \xi \geqslant \alpha|\xi|^p - |\delta(x)|, \end{cases}$$

(iv) a, b are strictly monotone, i.e.

$$\text{a.e. } \ x \in \Omega, \ \forall\, \xi, \eta \in \mathbb{R}^d, \ \xi \neq \eta, \quad \begin{cases} (a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0, \\ (b(x, \xi) - b(x, \eta)) \cdot (\xi - \eta) > 0. \end{cases}$$

In this nonlinear context we have the following result:

Theorem 2.19. *Let u_n and v_n be two sequences in $W_0^{1,p}(\Omega)$ such that*

$$u_n \rightharpoonup u \quad \textit{and} \quad v_n \rightharpoonup v \quad \textit{weakly in } W_0^{1,p}(\Omega), \tag{2.18}$$

$$-\operatorname{div}(a(x, \nabla u_n)) \geqslant 0 \quad \textit{and} \quad -\operatorname{div}(b(x, \nabla v_n)) \geqslant 0 \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.19}$$

Then we have the strong convergence

$$(v_n - v)^- \longrightarrow 0 \quad \textit{strongly in } W_0^{1,p}(\Omega), \tag{2.20}$$

and the following inequality holds true:

$$\forall \psi \in C^\infty(\overline{\Omega}),\ \ \psi \geqslant 0, \quad \liminf_{n\to+\infty} \int_\Omega \psi\, a(x, \nabla u_n)\cdot \nabla v_n \geqslant \int_\Omega \psi\, a(x, \nabla u)\cdot \nabla v. \tag{2.21}$$

Remark 2.20. As in Theorem 2.1 the operators defined by a and b are independent from each other (see also Remark 2.21 below).

Proof of Theorem 2.19.

Step 1: Proof of the strong convergence (2.20).

Only the proof of this step is rather different from the linear case and needs to be detailed.

Since $-\operatorname{div}(b(x,\nabla v_n)) \geqslant 0$, we have

$$\begin{aligned}
&\int_\Omega (b(x,\nabla v_n) - b(x,\nabla v))\cdot \nabla(v_n - v)\, \mathbf{1}_{\{v_n<v\}} \\
&\quad = -\int_\Omega (b(x,\nabla v_n) - b(x,\nabla v))\cdot \nabla(v_n - v)^- \\
&\quad = -\int_\Omega b(x,\nabla v_n)\cdot \nabla(v_n - v)^- + \int_\Omega b(x,\nabla v)\cdot \nabla(v_n - v)^- \\
&\quad \leqslant \int_\Omega b(x,\nabla v)\cdot \nabla(v_n - v)^-.
\end{aligned}$$

The right-hand side tends to 0 since $b(x,\nabla v)$ belongs to $L^{p'}(\Omega)^d$ while $(v_n - v)^-$ weakly converges to 0 in $W_0^{1,p}(\Omega)$. Therefore, by the monotonicity of b we obtain

$$\begin{aligned}
&e_n := (b(x,\nabla v_n) - b(x,\nabla v))\cdot \nabla(v_n - v)\, \mathbf{1}_{\{v_n<v\}} \geqslant 0 \quad \text{a.e. in } \Omega, \\
&\int_\Omega e_n \longrightarrow 0.
\end{aligned} \tag{2.22}$$

This convergence combined with the coerciveness of b implies the strong convergence (2.20): this is straightforward if one assumes that b is strongly monotone (and not only strictly monotone). In the general case, this is proved in the classical following way. Since the function e_n is nonnegative and strongly converges to 0 in $L^1(\Omega)$, there exists a subsequence n' of n and a measurable set E of zero Lebesgue measure such that for any $x \in \Omega \setminus E$,

$$\begin{cases} (\nabla(v_{n'} - v)^-)(x) = -(\nabla v_{n'} - \nabla v)(x)\, \mathbf{1}_{\{v_{n'}<v\}}(x), \\ \xi \longmapsto b(x,\xi) \quad \text{is continuous on} \quad \mathbb{R}^d, \\ e'_n(x) \underset{n\to+\infty}{\longrightarrow} 0. \end{cases}$$

Let us fix $x \in \Omega \setminus E$. Let n'' be a subsequence of n' (depending on x). If $v_{n''}(x) \geqslant v(x)$, then $(\nabla(v_{n''} - v))^-(x) = 0$. Otherwise (up to a new extraction), n'' is a subsequence such that $v_{n''}(x) < v(x)$. Then using coerciveness (iii) of b and the growth condition (ii), we deduce from

$$\begin{aligned} &\alpha|\nabla v''_n(x)|^p - |\delta(x)| \\ &\quad \leqslant b(x, \nabla v''_n(x)) \cdot \nabla v''_n(x) \\ &\quad = e''_n(x) + b(x, \nabla v''_n(x)) \cdot \nabla v(x) + b(x, \nabla v(x)) \cdot (\nabla v''_n(x) - \nabla v(x)) \\ &\quad \leqslant e''_n(x) + c\,(|\gamma(x)| + |\nabla v''_n(x)|^{p-1})|\nabla v(x)| + |b(x, \nabla v(x))| \left(|\nabla v''_n(x)| + |\nabla v(x)|\right) \end{aligned}$$

and from the convergence of $e'_n(x)$, that the sequence $\nabla v_{n''}(x)$ is bounded in $\mathbb{R}^d$ and thus converges (up to a new subsequence) to some $\xi_x \in \mathbb{R}^d$. Passing to the limit in $e''_n(x)$ thanks to the continuity of $b(x,\cdot)$ yields

$$(b(x, \xi_x) - b(x, \nabla v(x))) \cdot (\xi_x - \nabla v(x)) = 0,$$

whence $\xi_x = \nabla v(x)$ thanks to the strict monotonicity of b. Therefore, in view of the uniqueness of the limit we obtain for the whole sequence n''

$$\nabla(v_{n''} - v)^-(x) = -(\nabla v_{n''}(x) - \nabla v(x)) \underset{n\to+\infty}{\longrightarrow} 0.$$

In the two cases the sequence $\nabla(v_{n''} - v)^-(x)$ converges to 0 for any subsequence n'' of n' and hence $\nabla(v_{n'} - v)^-(x)$ converges to 0 for the whole sequence n'. This proves the almost everywhere convergence of $\nabla(v'_n - v_n))^-$ to 0.

On the other hand, we have by the coerciveness of b

$$\begin{aligned} &(\alpha\,|\nabla v_{n'}|^p - |\delta(x)|)\, \mathbf{1}_{\{v_{n'}<v\}} \\ &\quad \leqslant b(x, \nabla v_{n'}) \cdot \nabla v_{n'}\, \mathbf{1}_{\{v_{n'}<v\}} \\ &\quad = e_{n'}(x) + b(x, \nabla v_{n'}) \cdot \nabla v(x)\, \mathbf{1}_{\{v_{n'}<v\}} + b(x, \nabla v) \cdot (\nabla v_{n'} - \nabla v)\, \mathbf{1}_{\{v_{n'}<v\}}. \end{aligned}$$

The sequence $e_{n'}$ strongly converges to 0 in $L^1(\Omega)$ and the two other terms are equi-integrable in $L^1(\Omega)$; whence the sequence $|\nabla v_{n'}|^p\, \mathbf{1}_{\{v_{n'}<v\}}$ is equi-integrable in $L^1(\Omega)$ and so is $|\nabla v_{n'} - \nabla v|^p\, \mathbf{1}_{\{v_{n'}<v\}}$ since $|\nabla v_{n'} - \nabla v|^p \leqslant 2^p\,(|\nabla v_{n'}|^p + |\nabla v|^p)$.

In conclusion, the sequence $|\nabla(v_{n'} - v)^-|^p$ converges to 0 a.e. in Ω and is bounded and equi-integrable in $L^1(\Omega)$. Since its limit is 0 (and hence does not depend on the subsequence n'), the whole sequence $\nabla(v_n - v)^-$ strongly converges to 0 in $L^p(\Omega)^d$. This implies (2.20).

Step 2: Proof of (2.21).

To prove inequality (2.21) we proceed as in the linear case. The only difference is the fact that we should now prove the convergence

$$a(x, \nabla u_n) \rightharpoonup a(x, \nabla u) \quad \text{weakly in} \quad L^{p'}(\Omega)^d.$$

But this is a consequence of a compactness result of [5]: indeed, the assumptions on a, the weak convergence of u_n in $W_0^{1,p}(\Omega)$ and the positivity $-\operatorname{div}(a(x, \nabla u_n)) \geqslant 0$ imply by [5] that ∇u_n converges a.e. to ∇u in Ω. Then the boundedness of $a(x, \nabla u_n)$ in $L^{p'}(\Omega)^d$ and the continuity of $a(x, \cdot)$ immediately imply the desired convergence.

Remark 2.21. In contrast with the linear case, we used the coerciveness of a in step 2 of the proof of Theorem 2.19, only to prove that the sequence $a(\cdot, \nabla u_n)$ weakly converges to $a(\cdot, \nabla u)$.

3. The case of two varying operators

In this section, we consider the case of two operators $-\operatorname{div}(A_n \nabla)$ and $-\operatorname{div}(B_n \nabla)$ where A_n and B_n are two sequences of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ (see (2.1)), for given $\alpha, \beta > 0$, which H-converge in the sense of [22,25]. Let us recall this definition:

Definition 3.1. A sequence A_n of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ is said to H-converge to A in $\mathcal{M}(\alpha, \beta; \Omega)$, and we denote $A_n \xrightarrow{H} A$, if for any $f \in H^{-1}(\Omega)$, the solution u_n of

$$\begin{cases} -\operatorname{div}(A_n \nabla u_n) = f & \text{in} \quad \mathcal{D}'(\Omega), \\ u_n \in H_0^1(\Omega), \end{cases}$$

satisfies

$$u_n \rightharpoonup u \ \text{ weakly in } \ H_0^1(\Omega) \quad \text{and} \quad A_n \nabla u_n \rightharpoonup A \nabla u \ \text{ weakly in } \ L^2(\Omega)^d,$$

where u is the solution of

$$\begin{cases} -\operatorname{div}(A \nabla u) = f & \text{in } \mathcal{D}'(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$

Recall (see [22,25]) that from any sequence of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ one can extract a subsequence which H-converges.

In this context, we have the following result:

Theorem 3.2. *Consider two sequences of matrices A_n and B_n in $\mathcal{M}(\alpha, \beta; \Omega)$ such that*

$$A_n \xrightarrow{H} A. \tag{3.1}$$

Let u_n and v_n be two sequences in $H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u \quad \textit{and} \quad v_n \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega), \tag{3.2}$$

$$-\operatorname{div}(A_n \nabla u_n) \geqslant 0 \quad \textit{and} \quad -\operatorname{div}(B_n \nabla v_n) \geqslant 0 \quad \textit{in } \mathcal{D}'(\Omega). \tag{3.3}$$

Then for any function T such that

$$T \in C^1(\mathbb{R}), \quad \forall t \geqslant 0, \ T(t) = 0 \quad \textit{and} \quad \forall t \leqslant 0, \ -1 \leqslant T'(t) \leqslant 0, \tag{3.4}$$

we have the strong convergence

$$T(v_n - v) \longrightarrow 0 \quad \textit{strongly in } H_0^1(\Omega), \tag{3.5}$$

and the following inequality holds true:

$$\forall \psi \in C^\infty(\overline{\Omega}), \ \psi \geqslant 0, \quad \liminf_{n \to +\infty} \int_\Omega \psi \, A_n \nabla u_n \cdot \nabla v_n \geqslant \int_\Omega \psi \, A \nabla u \cdot \nabla v. \tag{3.6}$$

Remark 3.3. Theorem 3.2 does not need any assumption of H-convergence on the sequence B_n. If the sequence B_n is assumed to H-converge to some B, then one also has

$$\forall \psi \in C^\infty(\overline{\Omega}), \ \psi \geqslant 0, \quad \liminf_{n \to +\infty} \int_\Omega \psi \, \nabla u_n \cdot B_n \nabla v_n \geqslant \int_\Omega \psi \, \nabla u \cdot B \nabla v.$$

Remark 3.4. As in the case of two fixed operators (Section 2), the proof of Theorem 3.2 is not “symmetric" in u_n and v_n. Also, from the point of view of the homogenization theory, the proof of Theorem 3.2 is quite surprising since the so-called correctors, which measure the oscillations of ∇u_n and of ∇v_n (these sequences only converge weakly in $L^2(\Omega)^d$) do not appear in the proof.

Remark 3.5. As in the case of fixed operators, the strong convergence (3.5) and inequality (3.6) of Theorem 3.2 still hold true if the positivity assumptions (3.3) are replaced by

$$\begin{aligned} &-\operatorname{div}(A_n \nabla u_n) \geqslant F_n \quad \text{and} \quad -\operatorname{div}(B_n \nabla v_n) \geqslant G_n \quad \text{in } \mathcal{D}'(\Omega), \\ &\text{where} \quad F_n \longrightarrow F \quad \text{and} \quad G_n \longrightarrow G \quad \text{strongly in } H^{-1}(\Omega). \end{aligned}$$

Remark 3.6. In Theorem 3.2, it is not true that the sequence $(v_n - v)^-$ strongly converges to 0 in $H_0^1(\Omega)$ as in the case where B_n is fixed. Otherwise, any sequence v_n such that $-\operatorname{div}(B_n \nabla v_n)$ is compact in $H^{-1}(\Omega)$ would satisfy the following convergences: $(v_n - v)^-$ would strongly converge to 0 in $H_0^1(\Omega)$ using Remark 3.5, and $(v_n - v)^+$, which is equal to $(-v_n - (-v))^-$, would also strongly converge to 0 in $H_0^1(\Omega)$ since $-\operatorname{div}(B_n \nabla(-v_n))$ is also compact in $H^{-1}(\Omega)$. Hence $(v_n - v)$ would strongly converge to 0 in $H_0^1(\Omega)$, a result which is in general false in homogenization theory. The strong convergence (3.5) is therefore a weaker result than the strong convergence of $(v_n - v)^-$. This is the price to pay to take into account the oscillations of B_n.

However, assuming that the sequence B_n H-converges to some B and replacing the function v by the homogenization corrector $\bar{v}_n$ defined as in (4.5) below, one can prove that the sequence $(v_n - \bar{v}_n)^-$ strongly converges to 0 in $H_0^1(\Omega)$ (see Theorem 4.1 below).

Lemma 3.7. *Let A_n be a sequence of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ with $A_n \xrightarrow{H} A$ and let u_n be a sequence in $H_0^1(\Omega)$ such that*

$$u_n \rightharpoonup u \textit{ weakly in } H_0^1(\Omega) \quad \textit{and} \quad -\operatorname{div}(A_n \nabla u_n) \geqslant 0 \textit{ in } \mathcal{D}'(\Omega).$$

Then one has

$$A_n \nabla u_n \rightharpoonup A \nabla u \quad \textit{weakly in} \quad L^2(\Omega)^d.$$

For the sake of completeness, we will give below a proof of this lemma.

Proof of Theorem 3.2.

Step 1: Proof of the strong convergence (3.5).

We have

$$\int_\Omega B_n \nabla T(v_n - v) \cdot \nabla T(v_n - v) = \int_\Omega \big(T'(v_n - v)\big)^2 B_n \nabla(v_n - v) \cdot \nabla(v_n - v).$$

Since $-1 \leqslant T' \leqslant 0$, we have $(T')^2 \leqslant -T'$, whence

$$\begin{aligned}\int_\Omega B_n \nabla T(v_n - v) \cdot \nabla T(v_n - v) &\leqslant -\int_\Omega B_n \nabla(v_n - v) \cdot \nabla T(v_n - v)\\ &= -\int_\Omega B_n \nabla v_n \cdot \nabla T(v_n - v) + \int_\Omega T'(v_n - v)\, B_n \nabla v \cdot \nabla(v_n - v). \qquad (3.7)\end{aligned}$$

Using the uniform coerciveness of the matrices B_n, the facts that T and $-\operatorname{div}(B \nabla v_n)$ are nonnegative, Cauchy–Schwarz inequality and the boundedness of v_n in $H_0^1(\Omega)$, we deduce from (3.7) the inequality

$$\begin{aligned}\alpha \int_\Omega |\nabla T(v_n - v)|^2 &\leqslant \int_\Omega T'(v_n - v)\, B_n \nabla v \cdot \nabla(v_n - v)\\ &\leqslant \beta\, \|\nabla(v_n - v)\|_{L^2(\Omega)^d}\, \|T'(v_n - v)\nabla v\|_{L^2(\Omega)^d}\\ &\leqslant c\, \|T'(v_n - v)\nabla v\|_{L^2(\Omega)^d}.\end{aligned}$$

Since $(v_n - v)$ weakly converges to 0 in $H_0^1(\Omega)$, Rellich's compactness theorem implies that $(v_n - v)$ strongly converges to 0 in $L^2(\Omega)$ and (up to a subsequence) a.e. to 0 in Ω. Then by the continuity of T', $T'(v_n - v)$ also converges a.e. to 0 in Ω. Since $|T'(v_n - v)\nabla v| \leqslant |\nabla v|$ a.e. in Ω, Lebesgue's dominated convergence theorem yields that $T'(v_n - v)\nabla v$ strongly converges to 0 in $L^2(\Omega)^d$. Therefore, $T(v_n - v)$ strongly converges to 0 in $H_0^1(\Omega)$.

Step 2: Proof of inequality (3.6) for $\psi \in C_c^\infty(\Omega)$.

Let $\psi \in C_c^\infty(\Omega)$, $\psi \geqslant 0$. We have

$$\begin{aligned}\int_\Omega \psi\, A_n \nabla u_n \cdot \nabla v_n = \langle -\operatorname{div}(A_n \nabla u_n)\,,\, \psi(v_n - v)\rangle\\ + \int_\Omega A_n \nabla u_n \cdot \nabla(\psi v) - \int_\Omega v_n\, A_n \nabla u_n \cdot \nabla \psi,\end{aligned}$$

whence by Lemma 3.7 combined with the strong convergence of v_n in $L^2(\Omega)$

$$\begin{aligned}\int_\Omega \psi\, A_n \nabla u_n \cdot \nabla v_n &= \langle -\operatorname{div}(A_n \nabla u_n)\,,\, \psi(v_n - v)\rangle + \int_\Omega \psi\, A \nabla u \cdot \nabla v + o(1)\\ &\geqslant \langle -\operatorname{div}(A_n \nabla u_n)\,,\, -\psi(v_n - v)^-\rangle\\ &\quad + \int_\Omega \psi\, A \nabla u \cdot \nabla v + o(1).\end{aligned}$$

It thus remains to prove that

$$\langle -\operatorname{div}(A_n \nabla u_n)\,,\, -\psi(v_n - v)^-\rangle \underset{n\to+\infty}{\longrightarrow} 0 \tag{3.8}$$

in order to obtain inequality (3.6).

For that purpose, let us consider, for $\varepsilon > 0$, the function $T_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\begin{cases} T_\varepsilon(t) := 0 & \text{if } \; t \geqslant 0,\\ T_\varepsilon(t) := \frac{t^2}{2\varepsilon} & \text{if } \; -\varepsilon \leqslant t \leqslant 0,\\ T_\varepsilon(t) := -t - \frac{\varepsilon}{2} & \text{if } \; t \leqslant -\varepsilon.\end{cases}$$

Then $T_\varepsilon \in C^1(\mathbb{R})$ satisfies assumptions (3.4) of Theorem 3.2 and

$$\forall t \in \mathbb{R}, \quad |T_\varepsilon(t) - t^-| \leqslant \varepsilon.$$

Since $-\operatorname{div}(A_n \nabla u_n) \geqslant 0$ and since ψ belongs to $C_c^\infty(\Omega)$, we have

$$\begin{aligned} 0 \leqslant \langle -\operatorname{div}(A_n \nabla u_n), \psi(v_n - v)^- \rangle &\leqslant \langle -\operatorname{div}(A_n \nabla u_n), \psi\, T_\varepsilon(v_n - v) \rangle \\ &\quad + \varepsilon \langle -\operatorname{div}(A_n \nabla u_n), \psi \rangle. \end{aligned}$$

By the first step, the sequence $T_\varepsilon(v_n - v)$ strongly converges to 0 in $H_0^1(\Omega)$ for any fixed ε, while $-\operatorname{div}(A_n \nabla u_n)$ is bounded in $H^{-1}(\Omega)$, hence

$$0 \leqslant \langle -\operatorname{div}(A_n \nabla u_n), \psi(v_n - v)^- \rangle \leqslant o(1) + C\varepsilon,$$

which proves (3.8).

Step 3: Proof of inequality (3.6) for $\psi \in C^\infty(\overline{\Omega})$.

Let $\psi \in C^\infty(\overline{\Omega})$, $\psi \geqslant 0$, and let ψ_k be a sequence in $C_c^\infty(\Omega)$ such that $\psi_k(x)$ is nondecreasing and tends to $\psi(x)$ for any $x \in \Omega$. Therefore,

$$\langle -\operatorname{div}(A_n \nabla u_n), \psi v_n \rangle \geqslant \langle -\operatorname{div}(A_n \nabla u_n), \psi_k v_n \rangle, \tag{3.9}$$

whence using step 2 and the fact that

$$\int_\Omega v_n A_n \nabla u_n \cdot \nabla \psi_k \underset{n \to +\infty}{\longrightarrow} \int_\Omega v A \nabla u \cdot \nabla \psi_k \tag{3.10}$$

by Lemma 3.7, one has

$$\liminf_{n \to +\infty} \langle -\operatorname{div}(A_n \nabla u_n), \psi_k v_n \rangle \geqslant \langle -\operatorname{div}(A \nabla u), \psi_k v \rangle. \tag{3.11}$$

On the other hand, by Deny's theorem [15] (see also [26]) any function in $H_0^1(\Omega)$ has a quasi-continuous representative which belongs to $L^1(\mu)$ for any nonnegative measure μ which belongs to $H^{-1}(\Omega)$. Let $\mu := -\operatorname{div}(A \nabla u)$; then μ belongs to $H^{-1}(\Omega)$ and is nonnegative, and $\psi_k v \in H_0^1(\Omega)$ converges quasi-everywhere and hence μ-a.e. in Ω. Since $\psi_k v$ is a nondecreasing nonnegative sequence, Beppo–Levi's theorem implies that

$$\int_\Omega \psi_k v \, d\mu \underset{k \to +\infty}{\longrightarrow} \int_\Omega \psi v \, d\mu. \tag{3.12}$$

From (3.9), (3.11) and (3.12) we deduce that

$$\liminf_{n\to+\infty} \langle -\operatorname{div}(A_n\nabla u_n), \psi v_n\rangle \geqslant \langle -\operatorname{div}(A\nabla u), \psi v\rangle$$

and again using (3.10) with $\psi_k = \psi$, we obtain the desired inequality (3.6). □

Remark 3.8. In contrast with Theorem 2.1 where we do not need A to belong to $\mathcal{M}(\alpha, \beta; \Omega)$ but only $A \in L^\infty(\Omega)^{d\times d}$, we do need the assumptions $A_n \in \mathcal{M}(\alpha, \beta; \Omega)$ and $A_n \xrightarrow{H} A$ in Theorem 3.2. Indeed, even if these assumptions are not used in the above proof, we use them to prove Lemma 3.7 (see below).

Remark 3.9. Following the Remark 2.17 we consider here a nonnegative operator M in $\mathcal{L}(H_0^1(\Omega); H_0^1(\Omega)) \cap \mathcal{L}(L^\infty(\Omega); L^\infty(\Omega))$. Then, using in the proof of Theorem 3.2 the nonnegativity of the operator and the fact that $M \in \mathcal{L}(L^\infty(\Omega); L^\infty(\Omega))$ implies that

$$0 \leqslant M\left((v_n - v)^- - T_\varepsilon(v_n - v)\right) \leqslant C\varepsilon,$$

allows one to prove that under the assumptions of Theorem 3.2, one also has

$$\forall \psi \in C^\infty(\overline{\Omega}),\ \ \psi \geqslant 0, \quad \liminf_{n\to+\infty} \int_\Omega \psi\, A_n\nabla u_n \cdot \nabla(M v_n) \geqslant \int_\Omega \psi\, A\nabla u \cdot \nabla(M v).$$

Proof of Lemma 3.7. Let w_n be a corrector function in the sense of [22,25], associated with the matrix ${}^tA_n$ for the vector $\lambda \in \mathbb{R}^d$, i.e. a function which satisfies

$$w_n \rightharpoonup \lambda \cdot x \ \text{ weakly in } H^1(\Omega) \quad \text{and} \quad -\operatorname{div}\left({}^tA_n\nabla w_n\right) \ \text{ is compact in } H^{-1}(\Omega).$$

Since ${}^tA_n$ H-converges to ${}^tA$, the sequence ${}^tA_n\nabla w_n$ weakly converges to ${}^tA\lambda$ in $L^2(\Omega)^d$. Thanks to Meyers' L^p-regularity theorem [19], there exists $p > 2$ such that ∇w_n weakly converges to λ in $L^p_{\mathrm{loc}}(\Omega)^d$. On the other hand, by the compactness result of [24], $-\operatorname{div}(A_n\nabla u_n) \geqslant 0$ combined with the boundedness of $A_n\nabla u_n$ in $L^2(\Omega)^d$ implies that $-\operatorname{div}(A_n\nabla u_n)$ is actually compact in $W^{1,-p'}_{\mathrm{loc}}(\Omega)$ since $p' < 2$. Extract a subsequence (still denoted by n) such that

$$A_n\nabla u_n \rightharpoonup \sigma \quad \text{weakly in } L^2(\Omega)^d \quad \text{and} \quad L^{p'}_{\mathrm{loc}}(\Omega)^d,$$

the div–curl lemma in $L^{p'}_{\mathrm{loc}}(\Omega)^d \times L^p_{\mathrm{loc}}(\Omega)^d$ (see [23]) (or an integration by parts combined with Rellich's Theorem in $W_0^{1,p}(\Omega)$), implies that

$$A_n\nabla u_n \cdot \nabla w_n \rightharpoonup \sigma \cdot \lambda \quad \text{in} \quad \mathcal{D}'(\Omega).$$

On the other hand, the div–curl lemma in $L^2(\Omega)^d \times L^2(\Omega)^d$ (or again an integration by parts combined with Rellich's theorem in $H_0^1(\Omega)$) yields

$$A_n \nabla u_n \cdot \nabla w_n = \nabla u_n \cdot {}^t A_n \nabla w_n \rightharpoonup \nabla u \cdot {}^t A \lambda = A \nabla u \cdot \lambda \quad \text{in } \mathcal{D}'(\Omega).$$

Therefore $(A\nabla u - \sigma) \cdot \lambda = 0$ for any $\lambda \in \mathbb{R}^d$, whence $\sigma = A\nabla u$, which proves the lemma. □

4. The case of a sequence of quadratic forms

4.1. The general framework

Let H be a Hilbert space of real functions endowed with its norm $|\cdot|$ (in the applications, H will be $L^2(\Omega)$). Let V and V_n, $n \geqslant 1$, be Hilbert spaces endowed with their norm $\|\cdot\|$ and $\|\cdot\|_n$, respectively. Let $\dot{a}$ and $\dot{a}_n$, $n \geqslant 1$, be nonnegative quadratic forms whose domains (i.e., the set where the quadratic form is finite) are V and V_n, respectively. We assume that the following properties hold true:

- V is continuously and compactly imbedded in H and V_n is continuously imbedded in V, i.e.

$$\forall n \geqslant 1, \quad V_n \subset V \Subset H,$$

- $\left(u \longmapsto u^+\right)$ is weakly continuous from V into V and from V_n into V_n.
- the bilinear form $a : V \times V \longrightarrow \mathbb{R}$ associated with $\dot{a}$ is continuous and coercive, i.e. that there exist two positive constants α, β such that

$$\forall u, v \in V, \quad |a(u,v)| \leqslant \beta \, \|u\| \, \|v\| \quad \text{and} \quad \dot{a}(u) \geqslant \alpha \, \|u\|^2,$$

- the bilinear forms $a_n : V_n \times V_n \longrightarrow \mathbb{R}$ associated with $\dot{a}_n$ are continuous and coercive on V_n and are equi-coercive with respect to $\|\cdot\|$, i.e. that there exist constants $\gamma_n \geqslant 1$ such that

$$\forall u, v \in V_n, \quad |a_n(u,v)| \leqslant \gamma_n \, \|u\|_n \, \|v\|_n \quad \text{and} \quad \begin{cases} \dot{a}_n(u) \geqslant \gamma_n^{-1} \, \|u\|_n^2 \\ \dot{a}_n(u) \geqslant \alpha \, \|u\|^2, \end{cases}$$

- the bilinear forms a_n satisfy

$$\forall \, u \in V_n, \quad a_n(u^+, u^-) \leqslant 0,$$

- finally, the sequence $\dot{a}_n$ Γ-converges (see e.g. [12] for a general presentation of De Giorgi's Γ-convergence theory) to $\dot{a}$ for the strong topology of H, i.e.

$$\begin{cases} \forall \, u_n \longrightarrow u \ \text{ strongly in } \ H, & \dot{a}(u) \leqslant \liminf\limits_{n \to +\infty} \ \dot{a}_n(u_n), \\ \forall \, u \in H, \ \exists \, \tilde{u}_n \longrightarrow u \ \text{ strongly in } \ H, & \dot{a}(u) \geqslant \limsup\limits_{n \to +\infty} \ \dot{a}_n(\tilde{u}_n). \end{cases} \tag{4.1}$$

In this context, we have the following result:

Theorem 4.1. *Let u_n and v_n be two sequences in V_n such that*

$$u_n \rightharpoonup u \quad \textit{and} \quad v_n \rightharpoonup v \quad \textit{weakly in } V, \tag{4.2}$$

$$\textit{the sequence } \dot{a}_n(v_n) \textit{ is bounded}, \tag{4.3}$$

$$\forall\, w \in V_n,\ w \geqslant 0, \quad a(u_n, w) \geqslant 0 \quad \textit{and} \quad a(v_n, w) \geqslant 0. \tag{4.4}$$

Then there exists a unique $\bar{u}_n \in V_n$ such that

$$\forall\, w \in V_n, \quad a_n(\bar{u}_n, w) = a(u, w) \tag{4.5}$$

and $\bar{u}_n$ satisfies the following convergences:

$$\bar{u}_n \rightharpoonup u \quad \textit{weakly in } V \quad \textit{and} \quad (u_n - \bar{u}_n)^- \longrightarrow 0 \quad \textit{strongly in } V. \tag{4.6}$$

Moreover, the following inequality holds true:

$$\liminf_{n\to+\infty} a(u_n, v_n) \geqslant a(u, v). \tag{4.7}$$

Remark 4.2. Taking into account assumption (4.4), inequality (4.7) can be red as an extension of the lower semi-continuity inequality involving in definition (4.1) of the Γ-convergence. Indeed, (4.7) is the natural inequality deduced from (4.1) by polarization of the quadratic form $\dot{a}_n$ since a_n is the polar form associated with $\dot{a}_n$.

Remark 4.3. A sequence g_n in the dual space V_n' is said to be compact if

$$\forall\, w_n \in V_n, \qquad w_n \rightharpoonup 0 \text{ weakly in } V \quad \Longrightarrow \quad \langle g_n, w_n \rangle \underset{n\to+\infty}{\longrightarrow} 0. \tag{4.8}$$

If we replace the boundedness assumption (4.3) by compactness (4.8) of the sequence $a_n(u_n, \cdot)$ in the dual space V_n', we obtain that the sequence $(u_n - \bar{u}_n)$ strongly converges to 0 in V. So, $\bar{u}_n$ is a corrector of the sequence u_n in the Γ-convergence of $\dot{a}_n$.

Now, let us extend inequality (4.7) by introducing test functions. In view of that, we assume that there there exists a subset $\mathcal{D}$ of V such that, for any $\varphi \in \mathcal{D}$,

- the linear mapping $(u \longmapsto \varphi u)$ is defined from V into V and from V_n into V_n, and is sequentially weakly continuous on V, i.e.

$$\forall w_n \rightharpoonup w \quad \text{weakly in} \quad V, \quad \varphi w_n \rightharpoonup \varphi w \quad \text{weakly in} \quad V,$$

- there exists a positive constant c_φ (independent of n) such that

$$\forall w \in V_n, \quad \dot{a}_n(\varphi w) \leqslant c_\varphi\, \dot{a}_n(w). \tag{4.9}$$

Then under these extra assumptions we have the following result:

Theorem 4.4. *Let u_n and v_n be two sequences in V_n which satisfy assumptions* (4.2)–(4.4). *Let ψ be a function in $\mathcal{D}$ such that there exists g_n in the dual space V_n' which satisfies*

$$\forall w \in V_n,\ \ w \geqslant 0,\ \ a_n(\psi v_n, w) \geqslant \langle g_n, w\rangle \quad \textit{and} \quad g_n \textit{ is compact as in } (4.8). \tag{4.10}$$

Then the following extension of inequality (4.7) *holds true*

$$\liminf_{n\to+\infty}\ a_n(u_n, \psi v_n) \geqslant a(u, \psi v). \tag{4.11}$$

Remark 4.5. Let Ω be a bounded open set of $\mathbb{R}^d$, set $H := L^2(\Omega)$, $V_n = V := H_0^1(\Omega)$ and let $\dot{a}_n$ be a the quadratic form defined on $L^2(\Omega)$ by

$$\dot{a}_n(u) := \begin{cases} \int_\Omega A_n \nabla u \cdot \nabla u & \text{if} \quad u \in H_0^1(\Omega), \\ +\infty & \text{elsewhere}, \end{cases}$$

where A_n is a sequence of symmetric matrices in $\mathcal{M}(\alpha, \beta; \Omega)$, for given $\alpha, \beta > 0$, which H-converges (see Definition 3.1) to some A. In this context, the results of Theorem 3.2 with $B_n = A_n$ and Theorem 4.4 are the same since the H-convergence of A_n and the Γ-convergence of $\dot{a}_n$ are equivalent in the symmetric case.

Proof of Theorem 4.1.

Step 1: Proof of the convergences (4.6).

Thanks to the continuity and the coerciveness of a_n, Lax–Milgram's theorem implies that there exists a unique minimizer $\bar{u}_n$ of the functional $(\dot{a}_n - 2\,a(u,\cdot))$ defined on V_n and whose Euler's equation is given by (4.5). By the equi-coerciveness of a_n with respect to $\|\cdot\|$ and the continuity of a, we have

$$\alpha\,\|\bar{u}_n\|^2 \leqslant \dot{a}_n(\bar{u}_n) = a(u, \bar{u}_n) \leqslant \beta\,\|u\|\,\|\bar{u}_n\|,$$

whence $\bar{u}_n$ is bounded in V. Therefore, $\bar{u}_n$ weakly converges in V (up to a subsequence) to some $\bar{u} \in V$. Since a_n is equi-coercive with respect to $\|\cdot\|$, V is compactly imbedded in H and the linear operator $a(u,\cdot)$ is weakly continuous on V, the Γ-convergence (4.1) of $\dot{a}_n$ to $\dot{a}$ implies that $\bar{u}$ is the minimizer of the functional $(\dot{a} - 2\,a(u,\cdot))$ defined on V (this a classical property of the Γ-convergence which induces the convergence of the minima and the minimizers). Then by considering the Euler equation associated with $(\dot{a} - 2\,a(u,\cdot))$ we obtain $a(\bar{u},\cdot) = a(u,\cdot)$ in V, which yields $\bar{u} = u$. Therefore, the whole sequence $\bar{u}_n$ as u_n weakly converges to u in V.

By applying successively the inequality $a_n((u_n - \bar{u}_n)^+, (u_n - \bar{u}_n)^-) \leqslant 0$, the nonnegativity assumption (4.4) and definition (4.5) of $\bar{u}_n$, we have

$$\begin{aligned}\dot{a}_n\left((u_n - \bar{u}_n)^-\right) &\leqslant -a_n\left(u_n - \bar{u}_n, (u_n - \bar{u}_n)^-\right)\\ &= -a_n\left(u_n, (u_n - \bar{u}_n)^-\right) + a_n\left(\bar{u}_n, (u_n - \bar{u}_n)^-\right)\\ &\leqslant a_n\left(\bar{u}_n, (u_n - \bar{u}_n)^-\right) = a\left(u, (u_n - \bar{u}_n)^-\right).\end{aligned}$$

The last term tends to 0 by the continuity of a combined with the weak convergence of $(u_n - \bar{u}_n)^-$ to 0 in V (which holds by the weak continuity of $\left(u \longmapsto u^-\right)$ on V). Therefore, $\dot{a}_n\left((u_n - \bar{u}_n)^-\right)$ tends to 0 and, thanks to the equi-coerciveness of $\dot{a}_n$, $(u_n - \bar{u}_n)^-$ strongly converges to 0 in V.

Step 2: Proof of inequality (4.7).

By the positivity assumption (4.4) and definition (4.5) of $\bar{u}_n$ we have

$$\begin{aligned}a_n(u_n, v_n) &= a_n\left((u_n - \bar{u}_n)^+, v_n\right) - a_n\left((u_n - \bar{u}_n)^-, v_n\right) + a_n(\bar{u}_n, v_n)\\ &\geqslant -a_n\left((u_n - \bar{u}_n)^-, v_n\right) + a(u, v_n).\end{aligned} \tag{4.12}$$

Moreover, by the Cauchy–Schwarz inequality combined with boundedness (4.3) of $\dot{a}_n(v_n)$ we have

$$\left|a_n\left((u_n - \bar{u}_n)^-, v_n\right)\right| \leqslant \sqrt{\dot{a}_n(v_n)}\,\sqrt{\dot{a}_n\left((u_n - \bar{u}_n)^-\right)} \leqslant c\,\sqrt{\dot{a}_n\left((u_n - \bar{u}_n)^-\right)},$$

whence by the first step $a_n\left((u_n - \bar{u}_n)^-, v_n\right)$ tends to 0. Then, by passing to the lim inf in (4.12) and by using the continuity of a combined with the weak convergence of v_n in V, we obtain

$$\liminf_{n\to+\infty}\; a(u_n, v_n) \geqslant \liminf_{n\to+\infty}\; a(u, v_n) = a(u, v),$$

which concludes the proof. $\square$

Proof of Theorem 4.4. By (4.5), (4.10) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}a_n(u_n, \psi v_n) &= a_n\left((u_n - \bar{u}_n)^+, \psi v_n\right) - a_n\left((u_n - \bar{u}_n)^-, \psi v_n\right) + a_n(\bar{u}_n, v_n)\\ &\geqslant \langle g_n, (u_n - \bar{u}_n)^+\rangle - \sqrt{\dot{a}_n\left((u_n - \bar{u}_n)^-\right)}\,\sqrt{\dot{a}_n(\psi v_n)} + a(u, v_n).\end{aligned} \tag{4.13}$$

Since g_n is compact in V_n' according to (4.8) and since $(u_n - \bar{u}_n)^+$ weakly converges to 0 in V (by the weak continuity of $\left(u \longmapsto u^+\right)$ on V combined with the weak convergence of (4.6)), $\langle g_n, (u_n - \bar{u}_n)^+ \rangle$ tends to 0. Moreover, by (4.9) combined with (4.3) and by the result of step 1 we have

$$\dot{a}_n\left((u_n - \bar{u}_n)^-\right)\, \dot{a}_n(\psi v_n) \leqslant c_\psi\, \dot{a}_n(v_n)\, \dot{a}_n\left((u_n - \bar{u}_n)^-\right) \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore, passing to the lim inf in (4.13) yields

$$\liminf_{n\to+\infty} a(u_n, \psi v_n) \geqslant \liminf_{n\to+\infty} a(u, \psi v_n) = a(u, \psi v).$$

The last equality is a straightforward consequence of the continuity of a combined with the weak continuity of $(u \longmapsto \varphi u)$ on V. Theorem 4.4 is thus proved. $\square$

4.2. The case of varying domains

Let Ω be a bounded open set of $\mathbb{R}^d$. We consider an arbitrary sequence Ω_n of open subsets of Ω. Let $\dot{a}_n$ be the quadratic form defined on $L^2(\Omega)$ by

$$\dot{a}_n(u) := \begin{cases} \int_{\Omega_n} A\nabla u \cdot \nabla u & \text{if} \quad u \in H_0^1(\Omega_n) \text{ and } u = 0 \text{ in } \Omega \setminus \Omega_n, \\ +\infty & \text{elsewhere,} \end{cases} \tag{4.14}$$

where A is given symmetric matrix in $\mathcal{M}(\alpha, \beta; \Omega)$, for given $\alpha, \beta > 0$.

By [2,1,14,8] (see also [13] for the nonsymmetric case) the following compactness result holds true:

Theorem 4.6. *There exists a subsequence, still denoted by Ω_n, and a nonnegative Borel measure μ not loading the zero capacity sets such that $\dot{a}_n$ Γ-converges for the strong topology of $L^2(\Omega)$ to the quadratic form defined by*

$$\dot{a}(u) := \begin{cases} \int_{\Omega} A\nabla u \cdot \nabla u + \int_\Omega u^2\, d\mu & \textit{if} \quad u \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \\ +\infty & \textit{elsewhere.} \end{cases} \tag{4.15}$$

Several examples of sequences Ω_n with explicit measures μ are treated in [11].

Without loss of generality, we can assume that the result of Theorem 4.6 holds for the whole sequence. Then it is easy to check that the quadratic forms $\dot{a}$ (4.15) and $\dot{a}_n$ (4.14) satisfy the assumptions of Section 4.1. with the Hilbert spaces

$$H := L^2(\Omega), \quad V := H_0^1(\Omega) \quad \text{and} \quad V_n := \left\{ u \in H_0^1(\Omega_n) \,|\, u = 0 \text{ in } \Omega \setminus \Omega_n \right\}$$

endowed with the norms

$$|u| := \left(\int_\Omega u^2\right)^{1/2}, \quad \|u\| := \left(\int_\Omega |\nabla u|^2\right)^{1/2} \quad \text{and} \quad \|u\|_n := \left(\int_{\Omega_n} |\nabla u|^2\right)^{1/2}.$$

Moreover, we choose the set of "multipliers'' $\mathcal{D} := C^\infty(\overline{\Omega})$. Then, by the Poincaré inequality in Ω (with constant C_Ω) we have, for any $\varphi \in C^\infty(\overline{\Omega})$ and for any $w \in H_0^1(\Omega_n)$,

$$\begin{aligned} \dot{a}_n(\varphi w) &\leqslant \beta \int_\Omega |\nabla(\varphi w)|^2 \leqslant 2\,\beta \int_\Omega \left(|\nabla\varphi|^2 w^2 + \varphi^2|\nabla w|^2\right) \\ &\leqslant 2\,\beta\, \|\varphi\|^2_{W^{1,\infty}(\Omega)}\, \|w\|^2_{H^1(\Omega)} \leqslant 2\,\beta\, \|\varphi\|^2_{W^{1,\infty}(\Omega)}\, C_\Omega\, \|\nabla w\|^2_{L^2(\Omega)} \\ &\leqslant 2\,\frac{\beta}{\alpha}\, C_\Omega\, \|\varphi\|^2_{W^{1,\infty}(\Omega)} \int_\Omega A\nabla w \cdot \nabla w = c_\varphi\, \dot{a}_n(w), \end{aligned}$$

which thus yields condition (4.9).

Therefore, Theorem 4.4 applied to the sequence $\dot{a}_n$ defined by (4.14) implies the following result:

Theorem 4.7. *Let u_n and v_n be two sequences in V_n such that*

$$u_n \rightharpoonup u \quad \textit{and} \quad v_n \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega), \tag{4.16}$$

$$-\operatorname{div}(A\nabla u_n) \geqslant 0 \quad \textit{and} \quad -\operatorname{div}(A\nabla v_n) \geqslant 0 \quad \textit{in } \mathcal{D}'(\Omega_n). \tag{4.17}$$

Then the following inequality holds true:

$$\begin{aligned} &\forall\, \psi \in C^\infty(\overline{\Omega}),\ \psi \geqslant 0, \quad \liminf_{n\to+\infty} \int_{\Omega_n} \psi\, A\nabla u_n \cdot \nabla v_n \\ &\qquad \geqslant \int_\Omega \psi\, A\nabla u \cdot \nabla v + \int_\Omega \psi u v\, d\mu. \end{aligned} \tag{4.18}$$

Proof. First, note that the functions u and v defined by the weak convergences (4.16) belong to $L^2(\mu)$ since the lower semi-continuity property of the Γ-convergence implies that

$$\dot{a}(u) + \dot{a}(v) \leqslant \liminf_{n\to+\infty} (\dot{a}_n(u_n) + \dot{a}_n(v_n)) \leqslant \beta \liminf_{n\to+\infty} \int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) < +\infty,$$

hence u and v belong to the domain of $\dot{a}$ which is $H_0^1(\Omega) \cap L^2_\mu(\Omega)$.

On the one hand, the boundedness assumption (4.3) is clearly satisfied since

$$\dot{a}_n(v_n) \leqslant \beta \int_\Omega |\nabla v_n|^2 \leqslant c.$$

Let $\psi \in C_c^\infty(\Omega)$, $\psi \geqslant 0$. By definition (4.14) we have

$$a_n(\cdot, \psi v_n) = -\operatorname{div}(A\nabla(\psi v_n)) = -\psi \operatorname{div}(A\nabla v_n) - A\nabla v_n \cdot \nabla\psi - \operatorname{div}(v_n A\nabla\psi),$$

whence by (4.17) combined with $\psi \geqslant 0$

$$a_n(\cdot, \psi v_n) \geqslant g_n := -A\nabla v_n \cdot \nabla\psi - \operatorname{div}(v_n A\nabla\psi) \quad \text{in } \mathcal{D}'(\Omega_n),$$

where g_n is compact in the sense of (4.8) by Rellich's compactness theorem. Hence, assumption (4.10) is also satisfied.

On the other hand, we have by Rellich's compactness theorem

$$\begin{aligned}\int_{\Omega_n} \psi\, A\nabla u_n \cdot \nabla v_n &= a_n(u_n, \psi v_n) - \int_\Omega v_n\, A\nabla u_n \cdot \nabla\psi \\ &= a_n(u_n, \psi v_n) - \int_\Omega v\, A\nabla u \cdot \nabla\psi + o(1).\end{aligned}$$

Therefore, inequality (4.11) of Theorem 4.4 implies the desired inequality (4.18) since the polar form of $\dot{a}$ satisfies

$$a(u, \psi v) = \int_\Omega A\nabla u \cdot \nabla(\psi v) + \int_\Omega \psi u v\, d\mu.$$

Theorem 4.7 is proved. $\square$

4.3. The case of nonuniformly bounded operators

Let Ω be a bounded open set of $\mathbb{R}^d$, $d \geqslant 2$. We consider the case of a sequence of linear operators $-\operatorname{div}(A_n\nabla)$ where A_n is a sequence of symmetric matrices in $\mathcal{M}(\alpha, \beta_n; \Omega)$ with $\alpha > 0$ and $\beta_n \to +\infty$. So, the sequence $-\operatorname{div}(A_n\nabla)$ is equi-coercive but not equi-bounded. We associate to this operator the quadratic form defined on $L^2(\Omega)$ by

$$\dot{a}_n(u) := \begin{cases} \int_\Omega A_n\nabla u \cdot \nabla u & \text{if } \ u \in H_0^1(\Omega), \\ +\infty & \text{elsewhere.} \end{cases} \tag{4.19}$$

Since the first works of Khruslov [16,18], one knows that coefficients which are not equi-bounded can induce nonlocal effects in the limit process. In that sense and thanks to Beurling–Deny's representation theory of Dirichlet forms, Mosco [21, Theorem 4.1.2] proved the following asymptotic result:

Theorem 4.8 (*Mosco [21]*). *Assume that for any function $u \in C_0^1(\Omega)$, there exists a sequence $\bar{u}_n$* in *$L^2(\Omega)$ such that*

$$\bar{u}_n \longrightarrow u \ \ strongly\ in\ L^2(\Omega) \quad and \quad \liminf_{n\to+\infty} \dot{a}_n(\bar{u}_n) < +\infty. \tag{4.20}$$

Then there exist a subsequence, still denoted by $\dot{a}_n$, such that $\dot{a}_n$ Γ-converges for the strong topology of $L^2(\Omega)$ to the Dirichlet form $\dot{a}$ on $L^2(\Omega)$ whose restriction to $C_0^1(\Omega)$ is defined by

$$\begin{aligned}\dot{a}(u) = \int_\Omega A(dx)\nabla u \cdot \nabla u + \int_\Omega u^2\, k(dx)\\ + \int_{\Omega\times\Omega\setminus\mathrm{diag}} (u(x)-u(y))^2 \ j(dx,dy),\end{aligned} \tag{4.21}$$

where A is a nonnegative symmetric matrix-valued Radon measure on Ω, k a nonnegative Radon measure on Ω and j a nonnegative on $\Omega \times \Omega \setminus$ diag; *the three measures are uniquely determined by $\dot{a}$.*

The so-called jumping measure j in (4.21) expresses the appearance of nonlocal effects due to the unboundedness of A_n in (4.19). Various nonlocal effects have been obtained from suitable microstructures (like media reinforced by fibers) in [16,18,3,10,7,6].

In this context, we have the following result:

Theorem 4.9. *Let u_n and v_n be two sequences in $H_0^1(\Omega)$ such that*

$$u_n \rightharpoonup u \in C_0^1(\Omega) \quad and \quad v_n \rightharpoonup v \in C_0^1(\Omega) \quad weakly\ in\ H_0^1(\Omega), \tag{4.22}$$

$$\int_\Omega A_n \nabla v_n \cdot \nabla v_n \leqslant c, \tag{4.23}$$

$$-\operatorname{div}(A_n\nabla u_n) \geqslant 0 \quad \text{and} \quad -\operatorname{div}(A_n \nabla v_n) \geqslant 0 \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{4.24}$$

Then the following inequality holds true:

$$\begin{aligned}\liminf_{n\to+\infty} \int_\Omega A_n\nabla u_n \cdot \nabla v_n \geqslant \int_\Omega A(dx)\nabla u \cdot \nabla v + \int_\Omega uv\, k(dx)\\ + \int_{\Omega\times\Omega\setminus\mathrm{diag}} (u(x)-u(y))\\ \times (v(x)-v(y))\ j(dx,dy).\end{aligned} \tag{4.25}$$

Theorem 4.9 is a straightforward consequence of Theorems 4.1 and 4.8.

5. The case of higher-order operators

In this section, we restrict ourselves to the case $\Omega = \mathbb{R}^d$.

5.1. The case of fourth-order operators

We consider the second-order operator L defined on $\mathbb{R}^d$ by

$$Lu := -\operatorname{div}(A(x)\nabla u) - \operatorname{div}(a(x)u) + b(x)\cdot\nabla u + c(x)u, \quad u \in C_c^\infty(\mathbb{R}^d), \tag{5.1}$$

where A is a matrix in $\mathcal{M}(\alpha, \beta; \mathbb{R}^d)$ with $W^{2,\infty}(\mathbb{R}^d)$ coefficients, for given $\alpha, \beta > 0$, a and b are vectors with $W^{1,\infty}(\mathbb{R}^d)$ coefficients and $c \in W^{1,\infty}(\mathbb{R}^d)$. We denote by L^* the adjoint of the operator L, defined by

$$L^*u = -\operatorname{div}\left({}^tA(x)\nabla u\right) - \operatorname{div}(b(x)u) + a(x)\cdot\nabla u + c(x)u, \quad u \in C_c^\infty(\mathbb{R}^d).$$

We have the following semi-continuity result with the fourth operator L^*L:

Theorem 5.1. *Let u_n and v_n be two sequences in $H^2(\mathbb{R}^d)$ such that*

$$u_n \rightharpoonup u \quad \textit{and} \quad v_n \rightharpoonup v \quad \textit{weakly in } H^2(\mathbb{R}^d), \tag{5.2}$$

$$L^*Lu_n \geqslant 0 \quad \textit{and} \quad L^*Lv_n \geqslant 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^d). \tag{5.3}$$

Then we have the strong convergence

$$(Lu_n - Lu)^- \longrightarrow 0 \quad \textit{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^d) \tag{5.4}$$

and the following inequality holds true:

$$\forall\,\psi \in C_c^\infty(\mathbb{R}^d),\ \ \psi \geqslant 0, \quad \liminf_{n\to+\infty} \int_{\mathbb{R}^d} \psi\, Lu_n\, Lv_n \geqslant \int_{\mathbb{R}^d} \psi\, Lu\, Lv. \tag{5.5}$$

Under more restrictive assumptions on the operator L we can improve the result of Theorem 5.1:

Theorem 5.2. *Assume that the operator L is coercive, i.e that there exists a positive constant γ such that*

$$\forall\, w \in H^1(\mathbb{R}^d), \quad \langle Lw, w\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)} \geqslant \gamma\, \|u\|^2_{H^1(\mathbb{R}^d)}. \tag{5.6}$$

Then the strong convergence (5.4) *also holds in $L^2(\mathbb{R}^d)$ and inequality* (5.5) *also holds with $\psi = 1$.*

Remark 5.3. The proof of inequality (5.5) is quite different from the proofs done in Sections 2–4. Indeed, in the present context we cannot apply a truncature argument in the space $H^2(\mathbb{R}^d)$ since when u belongs to $H^2(\mathbb{R}^d)$, $|u|$ does not belong in general to $H^2(\mathbb{R}^d)$. Here, we will use the maximum principle applied to the operator $L + \lambda I$, for $\lambda > 0$ large enough, in Theorem 5.1 and to L in Theorem 5.2.

However, we cannot extend this approach to prove a semi-continuity result for the case of operators of order m with $m > 2$. We will therefore use another technique in Section 5.2.

Remark 5.4. In Theorem 5.2 the semi-continuity result holds in the whole space $\mathbb{R}^d$. We did not succeed in replacing $\mathbb{R}^d$ by any bounded (even smooth) open set Ω. Indeed, for a given function f in $L^2(\Omega)$, the unique solution $u \in H_0^1(\Omega)$ of $Lu + \lambda u = f$ in $\mathcal{D}'(\Omega)$ belongs to $H^2(\Omega)$ but not to $H_0^2(\Omega)$. Then boundary terms appear that we cannot eliminate. The situation is completely different in $\mathbb{R}^d$ since $H^2(\mathbb{R}^d) = H_0^2(\mathbb{R}^d)$.

We did not either succeed in extending inequality (5.5) to the case of any smooth function ψ with bounded derivatives of any order but whose support is not compact. Indeed, in our proof we use the compactness of the support of ψ to apply Rellich's compactness theorem.

Remark 5.5. As in Remarks 2.3 and 3.5, inequality (5.5) of Theorems 5.1 and 5.2 still holds true if the nonnegativity assumptions (5.3) are replaced by

$$\begin{array}{ll} & L^*Lu_n \geqslant F_n \quad \text{and} \quad L^*Lv_n \geqslant G_n \quad \text{in } \mathcal{D}'(\Omega), \\ \text{where} & F_n \longrightarrow F \quad \text{and} \quad G_n \longrightarrow G \quad \text{strongly in} \quad H^{-2}(\mathbb{R}^d). \end{array}$$

We will need the following result that we will prove below:

Lemma 5.6. *Let u_n be a sequence in $H^2(\mathbb{R}^d)$ such that*

$$u_n \rightharpoonup u \quad \textit{weakly in } H^2(\mathbb{R}^d) \quad \textit{and} \quad L^*Lu_n \geqslant 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^d). \tag{5.7}$$

Then we have up to a subsequence

$$Lu_n \longrightarrow Lu \quad \textit{a.e. in } \mathbb{R}^d. \tag{5.8}$$

Proof of 5.1.

Step 1: Proof of the strong convergence (5.4).

By definition (5.1) of the operator L, there exists a positive constant λ such that $(L + \lambda I)$ is H^1-coercive in the sense of (5.6). Since the bilinear form $(u, v) \longmapsto \langle Lu + \lambda u, v\rangle$ is continuous on $H^1(\mathbb{R}^d)^2$, and since $Lu_n - Lu$ belongs to $L^2(\mathbb{R}^d)$, Lax–Milgram lemma combined with the regularity result of solutions of second-order elliptic PDEs implies that there exists a unique solution q_n in $H^2(\mathbb{R}^d)$ of the equation

$$Lq_n + \lambda q_n = (Lu_n - Lu)^- \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d). \tag{5.9}$$

The sequence q_n is bounded in $H^2(\mathbb{R}^d)$ since Lu_n is bounded in $L^2(\mathbb{R}^d)$. By taking $\lambda > 0$ large enough, the maximum principle implies that $q_n \geqslant 0$ a.e. in $\mathbb{R}^d$. Moreover, by Lemma 5.6 $(Lu_n - Lu)^-$ tends to 0 a.e. in $\mathbb{R}^d$ up to a subsequence, hence the whole sequence $(Lu_n - Lu)^-$ weakly converges to 0 in $L^2(\mathbb{R}^d)$ (this a consequence of Egoroff's theorem). Therefore, the sequence q_n weakly tends to 0 in $H^2(\mathbb{R}^d)$.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geqslant 0$. Since by Rellich's compactness theorem q_n strongly converges to 0 in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ and since $(Lu_n - Lu)^-$ weakly converges to 0 in $L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi\,(Lu_n - Lu)^-\,(Lu_n - Lu)^- \\
&\quad = -\int_{\mathbb{R}^d} \varphi\,(Lu_n - Lu)\,(Lu_n - Lu)^- \\
&\quad = -\int_{\mathbb{R}^d} \varphi\,Lu_n\,(Lq_n + \lambda q_n) + \int_{\mathbb{R}^d} \varphi\,Lu\,(Lu_n - Lu)^- \\
&\quad = -\int_{\mathbb{R}^d} \varphi\,Lu_n\,Lq_n + o(1).
\end{aligned} \tag{5.10}$$

On the other hand, it is easy to check that $L(\varphi q_n) - \varphi L q_n$ has a compact support and is bounded in $H^1(\mathbb{R}^d)$, hence it is compact in $L^2(\mathbb{R}^d)$ by Rellich's compactness theorem. Therefore, since $L^* L u_n \geqslant 0$ and $\varphi q_n \geqslant 0$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi\,Lu_n\,Lq_n &= \int_{\mathbb{R}^d} Lu_n\,L(\varphi q_n) + o(1) \\
&= \langle L^* L u_n, \varphi q_n\rangle_{H^{-2}(\mathbb{R}^d), H^2(\mathbb{R}^d)} + o(1) \geqslant o(1).
\end{aligned} \tag{5.11}$$

Finally, estimate (5.10) combined with (5.11) yields

$$\int_{\mathbb{R}^d} \varphi\,(Lu_n - Lu)^-\,(Lu_n - Lu)^- \leqslant o(1),$$

which implies the strong convergence (5.4).

Step 2: Proof of inequality (5.5).

Let $\psi \in C_c^\infty(\mathbb{R}^d)$, $\psi \geqslant 0$. Let p_n be the unique solution in $H^1(\mathbb{R}^d)$ of the equation

$$L p_n + \lambda p_n = (Lu_n - Lu)^+ \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d). \tag{5.12}$$

As for the sequence q_n defined by (5.9), the sequence p_n weakly converges to 0 in $H^2(\mathbb{R}^d)$ and hence strongly converges to 0 in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ by Rellich's compactness

Theorem. By applying successively the strong convergence (5.4) of $(Lu_n - Lu)^-$ to 0 in $L^2_{\rm loc}(\mathbb{R}^d)$, the weak convergence of Lv_n to Lv in $L^2(\mathbb{R}^d)$ and the strong convergence of p_n to 0 in $H^1_{\rm loc}(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \psi\, Lu_n\, Lv_n \\
&\quad = \int_{\mathbb{R}^d} \psi\,(Lu_n - Lu)^+\, Lv_n - \int_{\mathbb{R}^d} \psi\,(Lu_n - Lu)^-\, Lv_n + \int_{\mathbb{R}^d} \psi\, Lu\, Lv_n \\
&\quad = \int_{\mathbb{R}^d} \psi\,(Lp_n + \lambda p_n)\, Lv_n + \int_{\mathbb{R}^d} \psi\, Lu\, Lv + o(1) \\
&\quad = \int_{\mathbb{R}^d} \psi\, Lp_n\, Lv_n + \int_{\mathbb{R}^d} \psi\, Lu\, Lv + o(1).
\end{aligned} \tag{5.13}$$

On the other hand, similarly to (φ, u_n, q_n) the triplet (ψ, v_n, p_n) satisfies estimate (5.11) which implies

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^d} \psi\, Lp_n\, Lv_n \geqslant 0.$$

This combined with (5.13) yields the desired inequality (5.5). $\square$

Proof of Theorem 5.2. Under the assumption of equi-coerciveness (5.6) we may consider the solution q_n of (5.9) with $\lambda = 0$. The maximum principle still holds, whence $q_n \geqslant 0$ a.e. in $\mathbb{R}^d$. Therefore, we can take $\varphi = 1$ in step 1 of the proof of Theorem 5.1. This implies the strong convergence of $(Lu_n - Lu)^-$ in $L^2(\mathbb{R}^d)$. Similarly, we can repeat step 2 of the proof of Theorem 5.1 in order to obtain inequality (5.5) with $\psi = 1$. $\square$

Proof of Lemma 5.6. By assumptions (5.2) and (5.3), L^*Lu_n is a nonnegative bounded sequence in $H^{-2}(\mathbb{R}^d)$. Then, by the compactness result of [24], L^*Lu_n is compact in $W^{-2,q}_{\rm loc}(\mathbb{R}^d)$ for any $q < 2$. For $q \in]1, 2[$ we set $q' := \frac{q}{q-1}$.

Let Ω be a smooth bounded open subset of $\mathbb{R}^d$ and let g_n be a sequence in $L^{q'}(\Omega)$ which weakly converges to 0 in $L^{q'}(\Omega)$. Using Meyers's L^p regularity result of solutions of second order elliptic PDE's, one can check that, for $\lambda > 0$ large enough, there exists $2 < q' < \frac{2d}{d-2}$ and a unique solution v_n in $W^{1,q'}_0(\Omega) \cap W^{2,q'}(\Omega)$ of the equation

$$Lv_n + \lambda v_n = g_n \quad \text{in} \quad \mathcal{D}'(\Omega).$$

Since g_n weakly converges to 0 in $L^{q'}(\Omega)$, the sequence v_n weakly converges to 0 in $W^{2,q'}_{\rm loc}(\Omega)$.

Let $\varphi \in C_c^\infty(\Omega)$. Since by Rellich's compactness theorem v_n strongly converges to 0 in $L^{q'}_{\mathrm{loc}}(\Omega)$, we have

$$\int_\Omega \varphi\, Lu_n\, g_n = \int_\Omega \varphi\, Lu_n\, Lv_n + \int_\Omega \lambda\, \varphi\, Lu_n\, v_n = \int_\Omega \varphi\, Lu_n\, Lv_n + o(1). \tag{5.14}$$

On the other hand, the sequence $L(\varphi v_n) - \varphi L v_n$ is bounded in $W_0^{1,q'}(\Omega)$ and is thus compact in $L^{q'}(\Omega)$. Therefore, since $L^* L u_n$ is compact in $W^{-2,q}_{\mathrm{loc}}(\mathbb{R}^d)$ and since φv_n weakly converges to 0 in $W_0^{2,q'}(\Omega)$, we have

$$\begin{aligned}\int_\Omega \varphi\, Lu_n\, Lv_n &= \int_\Omega Lu_n\, L(\varphi v_n) + o(1)\\ &= \langle L^* L u_n, \varphi v_n\rangle_{W^{-2,q}(\Omega), W^{2,q'}(\Omega)} + o(1) \underset{n\to+\infty}{\longrightarrow} 0.\end{aligned} \tag{5.15}$$

From (5.14) and (5.15) we thus deduce that

$$\int_\Omega \varphi\, L(u_n)\, g_n \underset{n\to+\infty}{\longrightarrow} 0$$

for any $\varphi \in C_c^\infty(\Omega)$ and for any sequence g_n weakly converging to 0 in $L^{q'}(\Omega)$. Therefore, Lu_n strongly converges to Lu in $L^q_{\mathrm{loc}}(\Omega)$, which implies that (up to a subsequence) Lu_n converges to Lu a.e. in Ω. Lemma 5.6 is proved. □

5.2. The case of a higher-order operator

As noted in Remark 5.3, it is difficult to extend the results of Section 5.1 obtained with any operator L of type (5.1), to a higher-order operator like L^*L for instance. Indeed, the proofs of Theorems 5.1 and 5.2 are based on the maximum principle satisfied by L but not by any higher-order operator. However, for very particular operators of higher order the maximum principle still holds true. So, we have the following result:

Theorem 5.7. *Let m be a positive integer. Let u_n and v_n be two sequences in $H^{2m}(\mathbb{R}^d)$ such that*

$$u_n \rightharpoonup u \quad \text{and} \quad v_n \rightharpoonup v \quad \text{weakly in } H^{2m}(\mathbb{R}^d), \tag{5.16}$$

$$(-\Delta)^{2m} u_n \geqslant 0 \quad \text{and} \quad (-\Delta)^{2m} v_n \geqslant 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{5.17}$$

Then for any $\psi \in C_c^\infty(\mathbb{R}^d)$, $\psi \geqslant 0$, the following inequality holds true:

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^d} \psi\, (-\Delta)^m u_n\, (-\Delta)^m v_n \geqslant \int_{\mathbb{R}^d} \psi\, (-\Delta)^m u\, (-\Delta)^m v. \tag{5.18}$$

We also have a global result if we replace $-\Delta$ by $(-\Delta + I)$:

Theorem 5.8. *Let m be a positive integer. Let u_n* and *v_n be two sequences in $H^{2m}(\mathbb{R}^d)$ such that*

$$u_n \rightharpoonup u \quad and \quad v_n \rightharpoonup v \quad weakly \ in \ H^{2m}(\mathbb{R}^d), \tag{5.19}$$

$$(-\Delta + I)^{2m} u_n \geqslant 0 \quad and \quad (-\Delta + I)^{2m} v_n \geqslant 0 \quad in \ \mathcal{D}'(\mathbb{R}^d). \tag{5.20}$$

Then the following inequality holds true:

$$\liminf_{n \to +\infty} \int_{\mathbb{R}^d} (-\Delta + I)^m u_n \, (-\Delta + I)^m v_n \geqslant \int_{\mathbb{R}^d} (-\Delta + I)^m u \, (-\Delta + I)^m v. \tag{5.21}$$

Remark 5.9. For $m = 1$ Theorems 5.7 and 5.8 are particular cases of Theorems 5.1 and 5.2 with $L = -\Delta$ and $L = -\Delta + I$, respectively.

Proof of Theorems 5.7 and 5.8 (Sketch). The proof is based on the argument used by Hedberg and the third author in [17]. Let G_m be the Bessel kernel defined by its Fourier transform:

$$\hat{G}_m(\xi) := (1 + |\xi|^2)^{-m}, \quad \text{for} \quad \xi \in \mathbb{R}^d.$$

The kernel G_m is a nonnegative function in $L^1(\mathbb{R}^d)$. Moreover, for any $p \in]1, +\infty[$ and for any function $H \in L^p(\mathbb{R}^d)$, the convolution $w := G_m * H$ is the unique solution of the equation

$$(-\Delta + I)^m w = H \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d).$$

Since G_m is nonnegative, the following maximum principle holds true:

$$H \geqslant 0 \ \text{a.e. in} \ \ \mathbb{R}^d \quad \Longrightarrow \quad w \geqslant 0 \ \text{a.e. in} \ \ \mathbb{R}^d. \tag{5.22}$$

On the other hand, by Calderón's result on singular integrals [9] we have

$$H \in L^p(\mathbb{R}^d) \quad \Longrightarrow \quad w \in W^{2m,p}(\mathbb{R}^d) \ \text{and} \ \|w\|_{W^{2m,p}(\mathbb{R}^d)} \leqslant C \, \|H\|_{L^p(\mathbb{R}^d)}, \tag{5.23}$$

where the constant C only depends on m and p.

Thanks to properties (5.22) and (5.23) we can repeat the proofs of Theorems 5.1 and 5.2 by replacing Eq. (5.9) by

$$(-\Delta + I)^m q_n = (Lu_n - Lu)^- \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d), \tag{5.24}$$

where $L := (-\Delta)^m$ for Theorem 5.7 and $L := (-\Delta + I)^m$ for Theorem 5.8. Note that the sequence q_n of (5.24) is bounded in $H^{2m}(\mathbb{R}^d)$ and satisfies the convergence

$$(-\Delta + I)^m q_n - (-\Delta)^m q_n \longrightarrow 0 \quad \text{strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^d). \qquad \square$$

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